ON THE MODULI SPACE OF SEMI-STABLE PLANE SHEAVES WITH HILBERT POLYNOMIAL $\mathrm{P}(\mathrm{m}) = 6\mathrm{m} + 2$

MARIO MAICAN

Abstract. We study the Simpson moduli space of semi-stable sheaves on the complex projective plane that have dimension 1, multiplicity 6 and Euler characteristic 2. We describe concretely these sheaves as cokernels of morphisms of locally free sheaves and we stratify the moduli space according to the types of sheaves that occur.

Contents

Acknowledgements. The author was supported by the Consiliul Naţional al Cercetării Ştiinţifice, an agency of the Romanian Government, grant PN II–RU 169/2010 PD–219.

1. Introduction

Let $\mathrm{M}_{\mathbb{P}^2}(r, \chi)$ denote the moduli space of Gieseker semi-stable sheaves on $\mathbb{P}^2(\mathbb{C})$ with Hilbert polynomial $\mathrm{P}(m) = rm + \chi$, r and χ being fixed integers, $r \geq 1$. Le Potier [6] found that $\mathrm{M}_{\mathbb{P}^2}(r, \chi)$ is an irreducible projective variety of dimension $r^2 + 1$, smooth at points given by stable sheaves and rational if $\chi \equiv 1$ or $2 \mod r$. In [2], [9] and [10] a complete description of semi-stable sheaves giving points in $\mathrm{M}_{\mathbb{P}^2}(4, \chi)$, $\mathrm{M}_{\mathbb{P}^2}(5, \chi)$ and $\mathrm{M}_{\mathbb{P}^2}(6, 1)$ was found. These moduli spaces were shown to have natural stratifications given by cohomological conditions on the sheaves involved. Here we are concerned with $\mathrm{M}_{\mathbb{P}^2}(6, 2)$. We describe all semi-stable sheaves giving points in $\mathrm{M}_{\mathbb{P}^2}(6, 2)$ and we decompose this moduli space into five strata: an open stratum X_0; a locally closed stratum that is the disjoint union of two irreducible locally closed subsets X_1 and X_2, each of codimension 3; a locally closed stratum that is the disjoint union of two irreducible locally closed subsets X_3 and X_4, each of codimension 5; an irreducible locally closed stratum X_5 of codimension 7 and a closed irreducible stratum X_6 of codimension 9. For some of these

1991 *Mathematics Subject Classification.* Primary 14D20, 14D22.

sets we have concrete geometric descriptions: X_1 is a certain open subset inside a fibre bundle with fibre $\mathbb{P}^{20}$ and base $\mathrm{N}(3,4,3) \times \mathbb{P}^2$, where $\mathrm{N}(3,4,3)$ is the moduli space of semi-stable Kronecker modules $f\colon 4\mathcal{O}(-2) \to 3\mathcal{O}(-1)$; X_3 is an open subset of a fibre bundle with fibre $\mathbb{P}^{22}$ and base $Y \times \mathrm{N}(3,2,3)$, where Y is the Hilbert scheme of zero-dimensional subschemes of $\mathbb{P}^2$ of length 2 and $\mathrm{N}(3,2,3)$ is the moduli space of semi-stable Kronecker modules $f\colon 2\mathcal{O}(-1) \to 3\mathcal{O}$; X_5 is an open subset of a fibre bundle with fibre $\mathbb{P}^{24}$ and base $\mathbb{P}^2 \times Y$; the closed stratum X_6 is isomorphic to the universal sextic in $\mathbb{P}^2 \times \mathbb{P}(\mathrm{S}^6 V^*)$. The following table contains a description of each X_i by cohomological conditions. The third column of the table lists all sheaves giving points in X_i. The sets W of morphisms φ are acted upon by the algebraic groups of automorphisms of sheaves and in each case, apart from X_0, the geometric quotient is X_i. The points given by properly semi-stable sheaves are all in X_0, which is why this stratum cannot be a geometric quotient of the set of morphisms. The table below is organised as the table in the introduction to [10], to which we generally refer for notations and conventions.

Let $C \subset \mathbb{P}^2$ denote an arbitrary smooth sextic curve and let P_i denote distinct points on C. The generic sheaves in X_1 are of the form $\mathcal{O}_C(1)(P_1 + \cdots + P_6 - P_7)$, where $P_1, \ldots, P_6$ are not contained in a conic curve. The generic sheaves in X_3 have the form $\mathcal{O}_C(2)(-P_1 - P_2 - P_3 + P_4 + P_5)$, where P_1, P_2, P_3 are non-colinear. The generic sheaves in X_4 are of the form $\mathcal{O}_C(1)(P_1 + \cdots + P_5)$, where $P_1, \ldots, P_5$ are in general linear position. The generic sheaves in X_5 are of the form $\mathcal{O}_C(2)(P_1 - P_2 - P_3)$. The sheaves giving points in X_6 are of the form $\mathcal{O}_C(2)(-P)$, (in this case C need not be smooth).

2. The open stratum

Proposition 2.1. *Every sheaf $\mathcal{F}$ giving a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfying the condition $\mathrm{h}^1(\mathcal{F}) = 0$ also satisfies the condition $\mathrm{h}^0(\mathcal{F}(-1)) = 0$. For these sheaves $\mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1)) = 0$ or 1. The sheaves from the first case are given by resolutions of the form*

$$0 \longrightarrow 4\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0, \tag{i}$$

where φ is not equivalent, modulo the action of the natural group of automorphisms, to a morphism represented by a matrix of the form

$$\begin{bmatrix} \star & 0 & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & \star \end{bmatrix}.$$

The sheaves in the second case are precisely the sheaves with resolution of the form

$$0 \longrightarrow 4\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0, \tag{ii}$$

where $\varphi_{12} = 0$, φ_{11} is semi-stable as a Kronecker V-module and φ_{22} has linearly independent entries.

Proof. The first statement follows from 6.4 [7]. The rest of the proposition follows by duality from 4.3 op.cit. $\square$

	cohomological conditions	W
X_0	$\mathrm{h}^0(\mathcal{F}(-1)) = 0$ $\mathrm{h}^1(\mathcal{F}) = 0$ $\mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1)) = 0$	$0 \longrightarrow 4\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$ φ is not equivalent to a morphism of any of the forms $\begin{bmatrix} \star & 0 & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix}, \quad \begin{bmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix}, \quad \begin{bmatrix} \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & \star \end{bmatrix}$
X_1	$\mathrm{h}^0(\mathcal{F}(-1)) = 0$ $\mathrm{h}^1(\mathcal{F}) = 0$ $\mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1)) = 1$	$0 \longrightarrow 4\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$ $\varphi_{12} = 0$, φ_{11} and φ_{22} are semi-stable as Kronecker modules
X_2	$\mathrm{h}^0(\mathcal{F}(-1)) = 0$ $\mathrm{h}^1(\mathcal{F}) = 1$ $\mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1)) = 1$	$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$ φ is not equivalent to a morphism of any of the forms $\begin{bmatrix} \star & \star & \star \\ \star & \star & 0 \\ \star & \star & 0 \end{bmatrix}, \quad \begin{bmatrix} \star & \star & \star \\ \star & 0 & \star \\ \star & 0 & \star \end{bmatrix}, \quad \begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & 0 & 0 \end{bmatrix}$
X_3	$\mathrm{h}^0(\mathcal{F}(-1)) = 0$ $\mathrm{h}^1(\mathcal{F}) = 1$ $\mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1)) = 2$	$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$ $\varphi_{13} = 0$, $\varphi_{12} \neq 0$ and does not divide φ_{11} φ_{23} has linearly independent maximal minors
X_4	$\mathrm{h}^0(\mathcal{F}(-1)) = 1$ $\mathrm{h}^1(\mathcal{F}) = 1$ $\mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1)) = 3$	$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \to 0$ φ is not equivalent to a morphism of any of the forms $\begin{bmatrix} \star & 0 & 0 \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}, \quad \begin{bmatrix} \star & \star & 0 \\ \star & \star & 0 \\ \star & \star & \star \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}, \quad \begin{bmatrix} 0 & \star & \star \\ 0 & \star & \star \\ \star & \star & \star \end{bmatrix}$
X_5	$\mathrm{h}^0(\mathcal{F}(-1)) = 1$ $\mathrm{h}^1(\mathcal{F}) = 2$ $\mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1)) = 4$	$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0$ φ_{11} has linearly independent entries $\varphi_{22} \neq 0$ and does not divide φ_{32}
X_6	$\mathrm{h}^0(\mathcal{F}(-1)) = 2$ $\mathrm{h}^1(\mathcal{F}) = 3$ $\mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1)) = 6$	$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O} \longrightarrow 2\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0$ φ_{12} has linearly independent entries

Let $\mathbb{W}_0 = \mathrm{Hom}(4\mathcal{O}(-2), 2\mathcal{O}(-1) \oplus 2\mathcal{O})$ and let $W_0 \subset \mathbb{W}_0$ be the set of morphisms φ from 2.1(i). Let

$$G_0 = (\mathrm{Aut}(4\mathcal{O}(-2)) \times \mathrm{Aut}(2\mathcal{O}(-1) \oplus 2\mathcal{O}))/\mathbb{C}^*$$

be the natural group acting by conjugation on $\mathbb{W}_0$. Let $X_0 \subset \mathrm{M}_{\mathbb{P}^2}(6,2)$ be the set of stable-equivalence classes of sheaves $\mathcal{F}$ as in 2.1(i). This set is open and dense.

Proposition 2.2. *There exists a categorical quotient of W_0 by G_0 and it is isomorphic to X_0.*

Proof. We have a canonical morphism $\rho\colon W_0 \to X_0$ mapping φ to the stable-equivalence class of $\mathcal{C}oker(\varphi)$. As at 4.2.1 [2], $\rho(\varphi_1) = \rho(\varphi_2)$ if and only if $\overline{G\varphi_1} \cap \overline{G\varphi_2} \neq \emptyset$. Thus any G_0-invariant morphism of varieties $f\colon W_0 \to Y$ factors through a unique map $g\colon X_0 \to Y$. To show that ρ is a categorical quotient map we use the method of 3.1.6 [2]. For any sheaf $\mathcal{F}$ giving a point in X_0 we need to obtain resolution 2.1(i) in a natural manner from the Beilinson spectral sequence converging to $\mathcal{F}$. We prefer, instead, to work with the Beilinson sequence of the dual sheaf $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$, which gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,4)$. Diagram (2.2.3) [2] takes the form

$$\begin{array}{ccc} 2\mathcal{O}(-2) & 0 & 0\ . \\ \\ 0 & 2\mathcal{O}(-1) & \xrightarrow{\varphi_4} 4\mathcal{O} \end{array}$$

The exact sequence (2.2.5) [2] takes the form

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{G} \longrightarrow 0.$$

According to (2.2.4) [2], φ_4 is injective. We now easily get the exact sequence dual to 2.1(i):

$$0 \longrightarrow 2\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow 4\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0. \qquad \square$$

Proposition 2.3. *If $\mathcal{F}$ is properly semi-stable and $\mathrm{P}_{\mathcal{F}}(t) = 6t + 2$, then $\mathcal{F}$ gives a point in X_0.*

3. The codimension 3 stratum

Let $\mathbb{W}_1 = \mathrm{Hom}(4\mathcal{O}(-2) \oplus \mathcal{O}(-1), 3\mathcal{O}(-1) \oplus 2\mathcal{O})$ and let $W_1 \subset \mathbb{W}_1$ be the set of morphisms φ from 2.1(ii). Let

$$G_1 = (\mathrm{Aut}(4\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \mathrm{Aut}(3\mathcal{O}(-1) \oplus 2\mathcal{O}))/\mathbb{C}^*$$

be the natural group acting by conjugation on $\mathbb{W}_1$. Let $X_1 \subset \mathrm{M}_{\mathbb{P}^2}(6,2)$ be the set of stable-equivalence classes of sheaves $\mathcal{F}$ as in 2.1(ii).

Proposition 3.1. *There exists a geometric quotient W_1/G_1 and it is a proper open subset inside a fibre bundle over $\mathrm{N}(3,4,3) \times \mathbb{P}^2$ with fibre $\mathbb{P}^{20}$. Moreover, W_1/G_1 is isomorphic to X_1. In particular, X_1 is irreducible and has codimension 3.*

Proof. The first statement can be proved identically as 2.2.2 [9]. Let W_1' be the locally closed subset of $\mathbb{W}_1$ given by the following conditions: $\varphi_{12} = 0$, φ_{11} is semi-stable as a Kronecker V-module, φ_{22} has linearly independent entries. Let $\Sigma \subset W_1'$ be the G_1-invariant subset given by the condition

$$\varphi_{21} = \varphi_{22}u + v\varphi_{11}, \quad u \in \mathrm{Hom}(4\mathcal{O}(-2), \mathcal{O}(-1)), \quad v \in \mathrm{Hom}(3\mathcal{O}(-1), 2\mathcal{O}).$$

As at loc.cit., we can construct a vector bundle Q over $\mathrm{N}(3,4,3) \times \mathbb{P}^2$ of rank 21 such that $\mathbb{P}(Q)$ is a geometric quotient of $W_1' \setminus \Sigma$ modulo G_1. Then W_1/G_1 is a proper open subset of $\mathbb{P}(Q)$.

Let $\mathcal{F}$ give a point in X_1 and let $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$. The Beilinson tableau (2.2.3) [2] for $\mathcal{G}$ takes the form

$$\begin{array}{ccc} 2\mathcal{O}(-2) \xrightarrow{\varphi_1} \mathcal{O}(-1) & & 0 \\ & & \\ 0 & & 3\mathcal{O}(-1) \xrightarrow{\varphi_4} 4\mathcal{O} \end{array} .$$

As in the proof of 2.2.3 [9], we have $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3)$ and an exact sequence

$$0 \longrightarrow \mathcal{O}(-3) \oplus 3\mathcal{O}(-1) \longrightarrow 4\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow \mathcal{C}oker(\varphi_1) \longrightarrow 0,$$

yielding the resolution

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus 3\mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus 4\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Since $\mathrm{H}^1(\mathcal{G}) = 0$, we see that $\mathcal{O}(-3)$ can be canceled yielding the dual of resolution 2.1(ii). □

Proposition 3.2. *The sheaves $\mathcal{G}$ from X_1^{D} are precisely the non-split extension sheaves of the form*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where $\mathbb{C}_x$ is the structure sheaf of a point $x \in \mathbb{P}^2$, $\mathcal{E}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,3)$ and satisfies the conditions $\mathrm{h}^0(\mathcal{E}(-1)) = 0$, $\mathrm{h}^1(\mathcal{E}) = 1$.

The generic sheaves $\mathcal{G}$ from X_1^{D} are of the form $\mathcal{O}_C(3)(-P_1 - \cdots - P_6 + P_7)$, where P_i are seven distinct points on a smooth sextic curve $C \subset \mathbb{P}^2$ and $P_1, \ldots, P_6$ are not contained in a conic curve.

By duality, the generic sheaves in X_1 are of the form $\mathcal{O}_C(1)(P_1 + \cdots + P_6 - P_7)$.

Proof. Assume that $\mathcal{G}$ gives a point in X_1^{D}, i.e. $\mathcal{G} \simeq \mathcal{C}oker(\varphi^{\mathrm{T}})$ for some morphism φ as in 2.1(ii). From the snake lemma we get an extension

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where x is the common zero of the entries of φ_{22} and $\mathcal{E}$ has a resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus 3\mathcal{O}(-1) \xrightarrow{\psi} 4\mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0,$$

$\psi_{12} = \varphi_{11}^{\mathrm{T}}$. From 5.3 [7] we know that $\mathcal{E}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,3)$ and satisfies the cohomological conditions from the proposition. Conversely, any such sheaf $\mathcal{E}$ is the cokernel of an injective morphism ψ for which ψ_{12} is semi-stable as a Kronecker V-module. Given a non-split extension of $\mathbb{C}_x$ by $\mathcal{E}$, we apply the horseshoe lemma to the above resolution

of $\mathcal{E}$ and to the standard resolution of $\mathbb{C}_x$ tensored with $\mathcal{O}(-1)$. The map $\mathcal{O}(-1) \to \mathbb{C}_x$ lifts to a map $\mathcal{O}(-1) \to \mathcal{G}$ because $\mathrm{H}^1(\mathcal{E}(1)) = 0$. We obtain a resolution

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus 3\mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus 4\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Since $\mathrm{Ext}^1(\mathbb{C}_x, 4\mathcal{O}) = 0$, we can deduce, as in the proof of 2.3.2 [9], that the morphism $\mathcal{O}(-3) \to \mathcal{O}(-3)$ is non-zero. We cancel $\mathcal{O}(-3)$ to get the dual to resolution 2.1(ii).

Let $X_{10} \subset X_1$ be the open subset of points given by sheaves $\mathcal{F} = \mathcal{C}oker(\varphi)$ for which the maximal minors of φ_{11} have no common factor. Let $X_{10}^{\mathrm{D}} \subset \mathrm{M}_{\mathbb{P}^2}(6,4)$ be the dual subset. According to [1], propositions 4.5 and 4.6, the sheaves $\mathcal{C}oker(\psi_{12})$, where the maximal minors of ψ_{12} have no common factor, are precisely the twisted ideal sheaves $\mathcal{I}_Z(3)$, where $Z \subset \mathbb{P}^2$ is a zero-dimensional scheme of length 6 not contained in a conic curve. It follows that the sheaves $\mathcal{G}$ giving points in X_{10}^{D} are precisely the non-split extensions of $\mathbb{C}_x$ by $\mathcal{J}_Z(3)$, where $\mathcal{J}_Z \subset \mathcal{O}_C$ is the ideal sheaf of a subscheme Z as above contained in a sextic curve C. Take C to be smooth and take Z to be the union of six distinct points different from x. Then $\mathcal{G} \simeq \mathcal{O}_C(3)(-P_1 - \cdots - P_6 + x)$. $\square$

Proposition 3.3. *Let $\mathcal{F}$ be a sheaf giving a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfying the conditions $\mathrm{h}^0(\mathcal{F}(-1)) = 0$, $\mathrm{h}^1(\mathcal{F}) = 1$. Then $\mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1)) = 1$ or 2. The sheaves in the first case are precisely the sheaves with resolution of the form*

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ is not equivalent to a morphism of any of the following forms:

$$\varphi_1 = \begin{bmatrix} \star & \star & \star \\ \star & \star & 0 \\ \star & \star & 0 \end{bmatrix}, \qquad \varphi_2 = \begin{bmatrix} \star & \star & \star \\ \star & 0 & \star \\ \star & 0 & \star \end{bmatrix}, \qquad \varphi_3 = \begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & 0 & 0 \end{bmatrix}.$$

Proof. Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfy the cohomological conditions from the hypothesis. Write $m = \mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1))$. As in the proof of 2.1.4 [9], the Beilinson free monad for $\mathcal{F}$ leads to a resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus m\mathcal{O}(-1) \xrightarrow{\varphi} (m-1)\mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

in which $\varphi_{13} = 0$. As $\mathcal{F}$ maps surjectively onto $\mathcal{C}oker(\varphi_{11}, \varphi_{12})$, we have $m \le 3$. If $m = 3$, then $\mathcal{C}oker(\varphi_{11}, \varphi_{12})$ has slope $-1/3$, so the semi-stability of $\mathcal{F}$ gets contradicted. Thus $m = 1$ or 2. Assume for the rest of this proof that $m = 1$. We have a resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

The conditions imposed on φ follow from the semi-stability of $\mathcal{F}$. Conversely, we assume that $\mathcal{F}$ has a resolution as in the proposition and we need to show that there are no destabilising subsheaves. Assume that $\mathcal{E} \subset \mathcal{F}$ is a destabilising subsheaf. We may take $\mathcal{E}$ to be semi-stable. As $\mathcal{F}$ is generated by global sections, we have $\mathrm{h}^0(\mathcal{E}) < \mathrm{h}^0(\mathcal{F})$. Thus $\mathcal{E}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(r,1)$ or $\mathrm{M}_{\mathbb{P}^2}(r,2)$ for some r, $1 \le r \le 5$. According to 2.3, the situation in which $\mathrm{P}_{\mathcal{E}}(t) = 3t+1$ is unfeasible. Moreover, we have $\mathrm{h}^0(\mathcal{E}(-1)) = 0$, $\mathrm{h}^0(\mathcal{E} \otimes \Omega^1(1)) \le 1$. From the results in [2] and [9] we see that $\mathcal{E}$ may have one of the following resolutions:

(1) $$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0, \tag{2}$$

$$0 \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0, \tag{3}$$

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0, \tag{4}$$

$$0 \longrightarrow 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0, \tag{5}$$

$$0 \longrightarrow 3\mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0, \tag{6}$$

$$0 \longrightarrow 3\mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow 2\mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0. \tag{7}$$

Each of these resolutions must fit into a commutative diagram like diagram (*) at 3.1 [10] in which α is injective on global sections. For the first four resolutions α must be injective and we get the contradictory conclusions that $\varphi \sim \varphi_1$, $\varphi \sim \varphi_2$ or $\varphi \sim \varphi_3$. If $\mathcal{E}$ has resolution (5), then β cannot be injective, hence α is not injective, hence $\mathcal{K}er(\alpha) \simeq \mathcal{K}er(\beta) \simeq \mathcal{O}(-1)$ and we conclude, as in the case of resolution (4), that $\varphi \sim \varphi_3$. If $\mathcal{E}$ has resolution (6), then, again, $\mathcal{K}er(\alpha) \simeq \mathcal{O}(-1) \simeq \mathcal{K}er(\beta)$, which is absurd, because $\mathcal{O}(-1)$ cannot be isomorphic to a subsheaf of $3\mathcal{O}(-2)$. For resolution (7) we arrive at a contradiction in a similar manner. $\square$

Let $\mathbb{W}_2 = \mathrm{Hom}(\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1), 3\mathcal{O})$ and let $W_2 \subset \mathbb{W}_2$ be the set of morphisms φ from proposition 3.3. Let

$$G_2 = (\mathrm{Aut}(\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \mathrm{Aut}(3\mathcal{O}))/\mathbb{C}^*$$

be the natural group acting by conjugation on $\mathbb{W}_2$. Let $X_2 \subset \mathrm{M}_{\mathbb{P}^2}(6, 2)$ be the set of stable-equivalence classes of sheaves of the form $\mathcal{C}oker(\varphi)$, $\varphi \in W_2$.

Proposition 3.4. *There exists a geometric quotient W_2/G_2, which is isomorphic to X_2. In particular, X_2 is irreducible and has codimension* 3.

Proof. Diagram (2.2.3) [2] for a sheaf $\mathcal{F}$ giving a point in X_2 takes the form

$$\begin{array}{ccccc} 4\mathcal{O}(-2) & \xrightarrow{\varphi_1} & 3\mathcal{O}(-1) & \xrightarrow{\varphi_2} & \mathcal{O} \\ & & & & \\ 0 & & \mathcal{O}(-1) & \xrightarrow{\varphi_4} & 3\mathcal{O} \end{array}$$

As in the proof of 2.2.4 [9], we may assume that φ_1 and φ_2 are given by

$$\varphi_1 = \begin{bmatrix} -Y & -Z & 0 & 0 \\ X & 0 & -Z & 0 \\ 0 & X & Y & 0 \end{bmatrix}, \qquad \varphi_2 = \begin{bmatrix} X & Y & Z \end{bmatrix}.$$

Thus $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3) \oplus \mathcal{O}(-2)$ and $\mathcal{I}m(\varphi_1) = \mathcal{K}er(\varphi_2)$. The exact sequence (2.2.5) [2] takes the form

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{F} \longrightarrow 0.$$

By 2.2 [2], φ_4 is injective. Clearly φ_5 lifts to a morphism $\varphi_5' \colon \mathcal{O}(-3) \oplus \mathcal{O}(-2) \to 3\mathcal{O}$. We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \begin{bmatrix} \varphi_5' & \varphi_4 \end{bmatrix}.$$

This proves that the map $W_2 \to X_2$ is a categorical quotient. According to [11], remark (2), p. 5, X_2 is normal. Applying [12], theorem 4.2, we conclude that the map $W_2 \to X_2$ is a geometric quotient. $\square$

4. The codimension 5 stratum

Proposition 4.1. *The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfying the cohomological conditions*

$$\mathrm{h}^0(\mathcal{F}(-1)) = 0, \qquad \mathrm{h}^1(\mathcal{F}) = 1, \qquad \mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1)) = 2$$

are precisely the sheaves with resolution of the form

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\varphi_{12} \neq 0$, $\varphi_{13} = 0$, φ_{11} is not divisible by φ_{12} and φ_{23} has linearly independent maximal minors.

Proof. At 3.3 we proved that a sheaf $\mathcal{F}$ giving a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfying the above cohomological conditions has a resolution as in the proposition. The conditions imposed on φ follow from the semi-stability of $\mathcal{F}$.

Conversely, assume that $\mathcal{F}$ has a resolution as in the proposition. Assume that there is a destabilising subsheaf $\mathcal{E} \subset \mathcal{F}$. We may assume that $\mathcal{E}$ is semi-stable. From the snake lemma we obtain an extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

where Z is the zero-dimensional scheme of length 2 given by the ideal $(\varphi_{11}, \varphi_{12})$ and $\mathcal{F}'$ has a resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-1) \xrightarrow{\psi} 3\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0$$

in which $\psi_{12} = \varphi_{23}$. According to 5.2 [10], $\mathcal{F}'$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,0)$ and the only subsheaf of $\mathcal{F}'$ of slope zero, if there is one, must be of the form $\mathcal{O}_L(-1)$ for a certain line $L \subset \mathbb{P}^2$. It follows that $\mathcal{E}$ must have Hilbert polynomial $\mathrm{P}_{\mathcal{E}}(t) = 2t+1$, $t+2$ or $t+1$. If $\mathrm{P}_{\mathcal{E}}(t) = 2t+1$, then $\mathcal{E}$ is the structure sheaf of some conic curve $C \subset \mathbb{P}^2$. We obtain a commutative diagram with exact rows and injective vertical maps

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-2) \arrow[r] \arrow[d, "\beta"] & \mathcal{O} \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}_C \arrow[r] \arrow[d] & 0 . \\
0 \arrow[r] & \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \arrow[r] & \mathcal{O}(-1) \oplus 3\mathcal{O} \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

Taking into account the possible canonical forms for β, we see that φ is represented by a matrix having one of the following forms:

$$\begin{bmatrix} \star & 0 & 0 & 0 \\ \star & 0 & \star & \star \\ \star & 0 & \star & \star \\ \star & \star & \star & \star \end{bmatrix}, \qquad \begin{bmatrix} \star & \star & 0 & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & \star \end{bmatrix}, \qquad \begin{bmatrix} \star & \star & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{bmatrix}.$$

In each of these situations the hypothesis on φ gets contradicted. If $\mathrm{P}_{\mathcal{E}}(t) = t + 1$, then $\mathcal{E}$ is the structure sheaf of some line $L \subset \mathbb{P}^2$ and we obtain a contradiction as above. The case in which $\mathrm{P}_{\mathcal{E}}(t) = t + 2$ is not feasible because in this case $\mathcal{E} \simeq \mathcal{O}_L(1)$, yet $\mathrm{H}^0(\mathcal{E}(-1))$ must vanish because the corresponding group for $\mathcal{F}$ vanishes. □

Let $\mathbb{W}_3 = \mathrm{Hom}(\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1), \mathcal{O}(-1) \oplus 3\mathcal{O})$ and let $W_3 \subset \mathbb{W}_3$ be the set of morphisms φ from proposition 4.1. Let

$$G_3 = (\mathrm{Aut}(\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1)) \times \mathrm{Aut}(\mathcal{O}(-1) \oplus 3\mathcal{O}))/\mathbb{C}^*$$

be the natural group acting by conjugation on $\mathbb{W}_3$. Let $X_3 \subset \mathrm{M}_{\mathbb{P}^2}(6, 2)$ be the set of stable-equivalence classes of sheaves of the form $\mathcal{C}oker(\varphi)$, $\varphi \in W_3$.

Proposition 4.2. *The generic sheaves in X_3 have the form $\mathcal{O}_C(2)(-P_1-P_2-P_3+P_4+P_5)$, where $C \subset \mathbb{P}^2$ is a smooth sextic curve, P_i are five distinct points on C and P_1, P_2, P_3 are non-colinear. In particular, X_3 lies in the closure of X_1. Moreover, X_3 also lies in the closure of X_2.*

Proof. Let $X_{30} \subset X_3$ be the open subset given by the following conditions: the equation $\det(\varphi) = 0$ determines a smooth sextic curve $C \subset \mathbb{P}^2$, the scheme Z from 4.1 consists of two distinct points P_4, P_5, the maximal minors of φ_{23} have no common factor and the subscheme $Y \subset \mathbb{P}^2$ they determine consists of three distinct points P_1, P_2, P_3, which are also distinct from P_4 and P_5. Let $\mathcal{F}$ give a point in X_{30}. According to 5.2 [10], the sheaf $\mathcal{F}'$ from 4.1 is isomorphic to $\mathcal{O}_C(2)(-P_1 - P_2 - P_3)$, hence $\mathcal{F}$ is isomorphic to $\mathcal{O}_C(-P_1 - P_2 - P_3 + P_4 + P_5)$. Conversely, we must show that any such sheaf $\mathcal{F}$ gives a point in X_{30}. We claim that $\mathcal{F}(1)$ has a global section that does not vanish at P_4 or P_5. The argument can be found at 2.3.2 [9] and it will be reproduced here for the sake of completeness. Let $\varepsilon_i\colon \mathrm{H}^0(\mathcal{O}_Z) \to \mathbb{C}$ be the linear form of evaluation at P_i, $i = 4, 5$. Let $\delta\colon \mathrm{H}^0(\mathcal{O}_Z) \to \mathrm{H}^1(\mathcal{O}_C(3)(-Y))$ be the connecting homomorphism arising from the exact sequence

$$0 \longrightarrow \mathcal{O}_C(3)(-Y) \longrightarrow \mathcal{F}(1) \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We must show that each ε_i is not orthogonal to $\mathrm{Ker}(\delta)$ or, which is the same, that each ε_i is not in the image of the dual map δ^*. By Serre duality δ^* is the restriction morphism

$$\mathrm{H}^0(\mathcal{O}_C(Y)) = \mathrm{H}^0(\mathcal{O}_C(-3)(Y) \otimes \omega_C) \longrightarrow \mathrm{H}^0((\mathcal{O}_C(-3)(Y) \otimes \omega_C)|_Z) = \mathrm{H}^0(\mathcal{O}_C(Y)|_Z).$$

We have the identity $\mathrm{H}^0(\mathcal{O}_C(Y)) = \mathrm{H}^0(\mathcal{O}_C) = \mathbb{C}$. This follows from the fact that the connecting homomorphism associated to the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(Y) \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

is injective. By Serre duality, this is equivalent to saying that the restriction morphism

$$\mathrm{H}^0(\mathcal{O}_C(3)) = \mathrm{H}^0(\mathcal{O}_C \otimes \omega_C) \longrightarrow \mathrm{H}^0((\mathcal{O}_C \otimes \omega_C)|_Y) = \mathrm{H}^0(\mathcal{O}_C(3)|_Y)$$

is surjective, and this is obvious. The claim now easily follows. We may now apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{O}_C(2)(-Y) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

and to the resolutions

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{I}_Z(2) \longrightarrow \mathcal{O}_C(2)(-Y) \longrightarrow 0.$$

Here $\mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^2}$ is the ideal sheaf of Y. We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{I}_Y(2) \longrightarrow \mathcal{F} \longrightarrow 0.$$

As in the proof of 2.3.2 [9], we can show that the morphism $\mathcal{O}(-4) \to \mathcal{O}(-4)$ above is non-zero. The argument uses the fact that $\mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{I}_Y(2)) = 0$. The vanishing of this group follows from the vanishing of $\mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Y)$ and of $\mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}(2))$, in view of the long Ext-sequence associated to the exact sequence

$$0 \longrightarrow \mathcal{I}_Y(2) \longrightarrow \mathcal{O}(2) \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Canceling $\mathcal{O}(-4)$ and taking into account that $\mathcal{I}_Y(2) \simeq \mathcal{C}oker(\psi)$ for some morphism $\psi\colon 2\mathcal{O}(-1) \to 3\mathcal{O}$ that is represented by a matrix with linearly independent maximal minors generating the ideal of Y (cf. the proof of 2.3.4(i) [9]), we obtain a resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

in which $\varphi_{13} = 0$, $\varphi_{23} = \psi$ and $\varphi_{11}, \varphi_{12}$ generate the ideal of Z. It is clear now that $\mathcal{F}$ gives a point in X_{30}.

To show that X_3 is included in $\overline{X}_1$ we choose a point in X_3 represented by the sheaf

$$\mathcal{O}_C(2)(-P_1 - P_2 - P_3 + P_4 + P_5).$$

We may assume that the line through P_1 and P_2 intersects C at six distinct points $P_1, P_2, Q_1, Q_2, Q_3, Q_4$, which are also distinct from P_4 and P_5. Then

$$\mathcal{O}_C(2)(-P_1 - P_2 - P_3 + P_4 + P_5) \simeq \mathcal{O}_C(1)(Q_1 + Q_2 + Q_3 + Q_4 - P_3 + P_4 + P_5).$$

Clearly, we can find points R_i on C converging to Q_i, $1 \le i \le 4$, which are distinct from P_3 and such that $R_1, R_2, R_3, R_4, P_4, P_5$ do not lie on a conic curve. According to 3.2, the sheaves

$$\mathcal{O}_C(1)(R_1 + R_2 + R_3 + R_4 - P_3 + P_4 + P_5)$$

represent points in X_1. These points converge to the chosen point in X_3. Thus $X_3 \subset \overline{X}_1$.

Taking into account the description, found at 3.3, of sheaves giving points in X_2, it is clear that for generic $\varphi \in W_3$ and for $t \in \mathbb{C}^*$ in a neighbourhood of zero the morphism $\varphi + t\pi$ is injective and its cokernel gives a point in X_2. Here π is projection onto the last component followed by injection into the first component. Clearly $[\mathcal{C}oker(\varphi + t\pi)]$ converges to $[\mathcal{C}oker(\varphi)]$ as t tends to 0. Thus $X_3 \subset \overline{X}_2$. □

Proposition 4.3. *There exists a geometric quotient W_3/G_3 and it is a proper open subset inside a fibre bundle with fibre $\mathbb{P}^{22}$ and base $Y \times \mathrm{N}(3,2,3)$, where Y is the Hilbert scheme of zero-dimensional subschemes of $\mathbb{P}^2$ of length 2. Moreover, W_3/G_3 is isomorphic to X_3.*

Proof. The construction of W_3/G_3 is identical to the construction of the quotient at 3.2.3 [9]. Let $W_3' \subset \mathbb{W}_3$ be the locally closed subset given by the conditions of 4.1, except injectivity. Let $\Sigma \subset W_3'$ be the G_3-invariant subset given by the condition

$$\varphi_{21} = \varphi_{22} u + v \varphi_{11}, \quad u \in \mathrm{Hom}(\mathcal{O}(-3) \oplus \mathcal{O}(-2), 2\mathcal{O}(-1)), \quad v \in \mathrm{Hom}(\mathcal{O}(-1), 3\mathcal{O}).$$

As at loc.cit., we can construct a vector bundle F over $Y \times \mathrm{N}(3,2,3)$ of rank 23 such that $\mathbb{P}(F)$ is a geometric quotient of $W_3' \setminus \Sigma$ modulo G_3. Then W_3/G_3 is a proper open subset of $\mathbb{P}(F)$.

Let $\mathcal{F}$ give a point in X_3. The Beilinson tableau (2.2.3) [2] for $\mathcal{F}$ has the form

$$\begin{array}{ccc} 4\mathcal{O}(-2) \xrightarrow{\varphi_1} & 4\mathcal{O}(-1) \xrightarrow{\varphi_2} & \mathcal{O} \quad . \\ & & \\ 0 & 2\mathcal{O}(-1) \xrightarrow{\varphi_4} & 3\mathcal{O} \end{array}$$

As at 6.5 [10], we have $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-4)$ and $\mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1) \simeq \mathcal{O}_Z$ for a scheme $Z \subset \mathbb{P}^2$ of dimension zero and length 2. The exact sequence (2.2.5) [2] takes the form

$$0 \longrightarrow \mathcal{O}(-4) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We claim that $\mathcal{F}(1)$ has a global section which maps to a global section of $\mathcal{O}_Z$ that generates this sheaf as an $\mathcal{O}_{\mathbb{P}^2}$-module. We have $\mathrm{h}^0(\mathcal{C}oker(\varphi_5)(1)) = 7$, $\mathrm{h}^0(\mathcal{F}(1)) = 8$, hence $\mathcal{F}(1)$ has a global section mapping to a non-zero section s of $\mathcal{O}_Z$. Consider an extension

$$0 \longrightarrow \mathbb{C}_{z_1} \longrightarrow \mathcal{O}_Z \longrightarrow \mathbb{C}_{z_2} \longrightarrow 0,$$

where z_1, z_2 are not necessarily distinct points in $\mathbb{P}^2$. If s maps to zero in $\mathbb{C}_{z_2}$, then s generates $\mathbb{C}_{z_1}$. Let $\mathcal{F}'$ be the preimage of $\mathbb{C}_{z_1}$ in $\mathcal{F}$. We apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{C}oker(\varphi_5) \longrightarrow \mathcal{F}' \longrightarrow \mathbb{C}_{z_1} \longrightarrow 0$$

and to the resolutions

$$0 \longrightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-1) \longrightarrow 3\mathcal{O} \longrightarrow \mathcal{C}oker(\varphi_5) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow 2\mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \longrightarrow \mathbb{C}_{z_1} \longrightarrow 0.$$

We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0$$

from which we get the relation $\mathrm{h}^0(\mathcal{F}') = 4$. This is absurd, $\mathrm{h}^0(\mathcal{F}')$ cannot exceed $\mathrm{h}^0(\mathcal{F})$. Thus the image of s in $\mathbb{C}_{z_2}$ is non-zero. When $z_1 = z_2$ this is enough to conclude that s generates $\mathcal{O}_Z$. When $z_1 \neq z_2$ we revert the roles of z_1 and z_2 in the above argument to deduce that s also does not vanish at z_1, so s generates $\mathcal{O}_Z$. We can now apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{C}oker(\varphi_5) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

to the above resolution of $\mathcal{C}oker(\varphi_5)$ and to the resolution

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-1) \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We obtain a resolution of the form

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow \mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Since $\mathrm{h}^1(\mathcal{F}) = 1$, the morphism $\mathcal{O}(-4) \to \mathcal{O}(-4)$ above is non-zero. Canceling $\mathcal{O}(-4)$ we arrive at resolution 4.1. In view of the method at 3.1.6 [2], we have proven that the canonical bijective map $W_3/G_3 \to X_3$ is an isomorphism. $\square$

Proposition 4.4. *The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfying the cohomological conditions $\mathrm{h}^0(\mathcal{F}(-1)) = 1$, $\mathrm{h}^1(\mathcal{F}) = 1$ are precisely the sheaves having resolution of the form*

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ is not equivalent to a morphism represented by a matrix having one of the following forms:

$$\left[\begin{array}{ccc} \star & 0 & 0 \\ \star & \star & \star \\ \star & \star & \star \end{array}\right], \quad \left[\begin{array}{ccc} \star & \star & 0 \\ \star & \star & 0 \\ \star & \star & \star \end{array}\right], \quad \left[\begin{array}{ccc} 0 & 0 & \star \\ \star & \star & \star \\ \star & \star & \star \end{array}\right], \quad \left[\begin{array}{ccc} 0 & \star & \star \\ 0 & \star & \star \\ \star & \star & \star \end{array}\right].$$

Proof. Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfy the cohomological conditions from the proposition. Write $m = \mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1))$. The Beilinson free monad (2.2.1) [2] for $\mathcal{F}$ reads

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow 5\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \longrightarrow (m+2)\mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{O} \longrightarrow 0$$

and gives the resolution

$$0 \longrightarrow \mathcal{O}(-2) \longrightarrow 5\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \longrightarrow \Omega^1 \oplus (m-1)\mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Using the Euler sequence and arguing as at 2.1.4 [9] we arrive at a resolution

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\eta} \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \xrightarrow{\varphi} (m-1)\mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\eta = \left[\begin{array}{c} 0 \\ 0 \\ \eta_{31} \end{array}\right], \qquad \varphi = \left[\begin{array}{ccc} \varphi_{11} & \varphi_{12} & 0 \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \end{array}\right].$$

As at loc.cit., the entries of η_{31} span V^*, hence $m \geq 3$. From the fact that $\mathcal{F}$ maps surjectively onto $\mathcal{C}oker(\varphi_{11}, \varphi_{12})$ we get the reverse inequality. Thus $m = 3$, $\mathcal{C}oker(\eta_{31}) \simeq \Omega^1(1)$ and we have a resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \oplus \Omega^1(1) \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus 3\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

in which $\varphi_{13} = 0$. Arguing as at loc.cit., we can show that $\mathcal{C}oker(\varphi_{23}) \simeq \mathcal{O}(1)$, so we arrive at a resolution as in the proposition. The conditions imposed on φ follow from the semi-stability of $\mathcal{F}$.

Conversely, we assume that $\mathcal{F}$ has a resolution as in the proposition and we must show that there are no destabilising subsheaves. Write

$$\psi = \left[\begin{array}{cc} \varphi_{11} & \varphi_{12} \end{array}\right] = \left[\begin{array}{ccc} q_1 & \ell_{11} & \ell_{12} \\ q_2 & \ell_{21} & \ell_{22} \end{array}\right].$$

As noted at 4.1 [10], the conditions on φ in the proposition are equivalent to saying that

$$\left|\begin{array}{cc} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{array}\right| \neq 0 \quad \text{and} \quad \left|\begin{array}{cc} q_1 & \ell_{11} \\ q_2 & \ell_{21} \end{array}\right|, \quad \left|\begin{array}{cc} q_1 & \ell_{12} \\ q_2 & \ell_{22} \end{array}\right|$$

are linearly independent in $\mathrm{S}^3\, V^*/(\ell_{11}\ell_{22} - \ell_{12}\ell_{21})V^*$. Thus the maximal minors of ψ cannot have a quadratic common factor. It follows that $\mathcal{K}er(\psi) \simeq \mathcal{O}(-4)$, if the maximal minors of ψ have a linear common factor, or $\mathcal{K}er(\psi) \simeq \mathcal{O}(-5)$, if they have no common factor. From the snake lemma we have an exact sequence

$$0 \longrightarrow \mathcal{K}er(\psi) \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}oker(\psi) \longrightarrow 0.$$

Assume that $\mathcal{K}er(\psi) \simeq \mathcal{O}(-4)$. Because of the conditions on ψ it is easy to check that $\mathcal{C}oker(\psi)$ has zero-dimensional torsion of length at most 1. Assume that $\mathcal{C}oker(\psi)$ has no zero-dimensional torsion. Then $\mathcal{C}oker(\psi) \simeq \mathcal{O}_L(1)$ for a line $L \subset \mathbb{P}^2$ and we have an extension

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_L(1) \longrightarrow 0,$$

where $C \subset \mathbb{P}^2$ is a quintic curve. Let $\mathcal{F}' \subset \mathcal{F}$ be a non-zero subsheaf of multiplicity at most 5. Denote by $\mathcal{C}$ its image in $\mathcal{O}_L(1)$ and put $\mathcal{K} = \mathcal{F}' \cap \mathcal{O}_C(1)$. Let $\mathcal{A}$ be a sheaf as in 3.1.2 [9]. If $\mathcal{C} = 0$, then $\mathrm{p}(\mathcal{F}') \le 0$ because $\mathcal{O}_C(1)$ is stable. We may, therefore, assume that $\mathcal{C} \neq 0$. We can estimate the slope of $\mathcal{F}'$ as at loc.cit.:

$$\begin{aligned}
\mathrm{P}_{\mathcal{F}'}(t) &= \mathrm{P}_{\mathcal{K}}(t) + \mathrm{P}_{\mathcal{C}}(t)\\
&= \mathrm{P}_{\mathcal{A}}(t) - \mathrm{h}^0(\mathcal{A}/\mathcal{K}) + \mathrm{P}_{\mathcal{O}_L(1)}(t) - \mathrm{h}^0(\mathcal{O}_L(1)/\mathcal{C})\\
&= (5-d)t + \frac{d^2-5d}{2} + t + 2 - \mathrm{h}^0(\mathcal{A}/\mathcal{K}) - \mathrm{h}^0(\mathcal{O}_L(1)/\mathcal{C}),
\end{aligned}$$

where d is an integer, $1 \le d \le 4$. Thus

$$\mathrm{p}(\mathcal{F}') = \frac{1}{6-d}\left(\frac{d^2-5d}{2} + 2 - \mathrm{h}^0(\mathcal{A}/\mathcal{K}) - \mathrm{h}^0(\mathcal{O}_L(1)/\mathcal{C})\right) \le \frac{d^2-5d+4}{2(6-d)} < \frac{1}{3} = \mathrm{p}(\mathcal{F}).$$

We see that in this case $\mathcal{F}$ is stable. Assume next that $\mathcal{C}oker(\psi)$ has a zero-dimensional subsheaf $\mathcal{T}$ of length 1. Let $\mathcal{E}$ be the preimage of $\mathcal{T}$ in $\mathcal{F}$. According to 3.1.5 [9], $\mathcal{E}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(5,1)$. Let $\mathcal{F}'$ and $\mathcal{C}$ be as above. If $\mathcal{C} \subset \mathcal{T}$, then $\mathcal{F}' \subset \mathcal{E}$, hence $\mathrm{p}(\mathcal{F}') \le \mathrm{p}(\mathcal{E}) < \mathrm{p}(\mathcal{F})$. If $\mathcal{C}$ is not a subsheaf of $\mathcal{T}$, then we can estimate the slope of $\mathcal{F}'$ as above concluding again that it is less than the slope of $\mathcal{F}$.

Assume now that $\mathcal{K}er(\psi) \simeq \mathcal{O}(-5)$. We have an extension

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{T} \longrightarrow 0,$$

where $C \subset \mathbb{P}^2$ is a sextic curve and $\mathcal{T}$ is a zero-dimensional sheaf of length 5. Let $\mathcal{F}' \subset \mathcal{F}$ be a subsheaf of multiplicity at most 5, let $\mathcal{T}'$ be its image in $\mathcal{T}$ and put $\mathcal{K} = \mathcal{F}' \cap \mathcal{O}_C(1)$. As above, we have

$$\begin{aligned}
\mathrm{P}_{\mathcal{F}'}(t) &= \mathrm{P}_{\mathcal{K}}(t) + \mathrm{h}^0(\mathcal{T}')\\
&= \mathrm{P}_{\mathcal{A}}(t) - \mathrm{h}^0(\mathcal{A}/\mathcal{K}) + \mathrm{h}^0(\mathcal{T}')\\
&= (6-d)t + \frac{d^2-5d-6}{2} - \mathrm{h}^0(\mathcal{A}/\mathcal{K}) + \mathrm{h}^0(\mathcal{T}'),\\
\mathrm{p}(\mathcal{F}') &= -\frac{d+1}{2} + \frac{\mathrm{h}^0(\mathcal{T}') - \mathrm{h}^0(\mathcal{A}/\mathcal{K})}{6-d} \le -\frac{d+1}{2} + \frac{5}{6-d},
\end{aligned}$$

where d is an integer, $1 \le d \le 5$. We see from this that $\mathrm{p}(\mathcal{F}') < \mathrm{p}(\mathcal{F})$ except, possibly, when $d = 5$ and $\mathrm{P}_{\mathcal{F}'}(t) = t+1$ or $t+2$, i.e. when $\mathcal{F}'$ is isomorphic to $\mathcal{O}_L$ or $\mathcal{O}_L(1)$ for a line $L \subset \mathbb{P}^2$. These situations can easily be ruled out. If, say, $\mathcal{O}_L$ were a subsheaf of $\mathcal{F}$,

then we would get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-1) \arrow[r] \arrow[d, "\beta"] & \mathcal{O} \arrow[r] \arrow[d, "\alpha"] & \mathcal{O}_L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \arrow[r] & 2\mathcal{O}(-1) \oplus \mathcal{O}(1) \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

in which α is injective, because it is injective on global sections. Thus β is also injective, which is absurd. We conclude that $\mathcal{F}$ is stable. □

Let $\mathbb{W}_4 = \text{Hom}(\mathcal{O}(-3) \oplus 2\mathcal{O}(-2), 2\mathcal{O}(-1) \oplus \mathcal{O}(1))$ and let $W_4 \subset \mathbb{W}_4$ be the set of morphisms φ from proposition 4.4. Let

$$G_4 = (\text{Aut}(\mathcal{O}(-3) \oplus 2\mathcal{O}(-2)) \times \text{Aut}(2\mathcal{O}(-1) \oplus \mathcal{O}(1)))/\mathbb{C}^*$$

be the natural group acting by conjugation on $\mathbb{W}_4$. Let $X_4 \subset \text{M}_{\mathbb{P}^2}(6,2)$ be the set of stable-equivalence classes of sheaves of the form $\mathcal{C}oker(\varphi)$, $\varphi \in W_4$.

Proposition 4.5. *There exists a geometric quotient W_4/G_4, which is isomorphic to X_4. In particular, X_4 is irreducible and has codimension 5.*

Proof. The Beilinson diagram (2.2.3) [2] for the dual sheaf $\mathcal{G} = \mathcal{F}^{\text{D}}(1)$ giving a point in $\text{M}_{\mathbb{P}^2}(6,4)$ has the form

$$3\mathcal{O}(-2) \xrightarrow{\varphi_1} 3\mathcal{O}(-1) \xrightarrow{\varphi_2} \mathcal{O} \quad .$$

$$\mathcal{O}(-2) \xrightarrow{\varphi_3} 5\mathcal{O}(-1) \xrightarrow{\varphi_4} 5\mathcal{O}$$

As in the proof of 2.2.4 [9], we have $\mathcal{K}er(\varphi_2) = \mathcal{I}m(\varphi_1)$ and $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3)$. Combining the exact sequences (2.2.4) and (2.2.5) [2] we get the resolution

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-3) \oplus 5\mathcal{O}(-1) \longrightarrow 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.$$

As in the proof of 2.1.4 [9], we have $\mathcal{C}oker(\psi) \simeq \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \oplus \Omega^1(1)$. We get the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \oplus \Omega^1(1) \xrightarrow{\varphi} 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.$$

As at loc.cit., we have $\mathcal{C}oker(\varphi_{13}) \simeq 2\mathcal{O} \oplus \mathcal{O}(1)$. We finally arrive at the resolution dual to resolution 4.4:

$$0 \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-1) \longrightarrow 2\mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{G} \longrightarrow 0.$$

This proves that the map $W_4 \to X_4$ is a categorical quotient. According to [11], remark (2), p. 5, X_4 is normal. Applying [12], theorem 4.2, we conclude that the map $W_4 \to X_4$ is a geometric quotient. □

Proposition 4.6. *The generic sheaves in X_4 are of the form $\mathcal{O}_C(1)(P_1 + \cdots + P_5)$, where $C \subset \mathbb{P}^2$ is a smooth sextic curve and P_i are five distinct points on C, no three of which are colinear. In particular, X_4 lies in the closure of X_1.*

Proof. Let $X_{40} \subset X_4$ be the subset defined by the following conditions: the sextic curve C given by the equation $\det(\varphi) = 0$ is smooth, the conic curve F given by the equation $f = \ell_{11}\ell_{22} - \ell_{12}\ell_{21} = 0$ is irreducible, there are constants $c_1, c_2 \in \mathbb{C}$ such that the cubic curve G with equation

$$c_1 \begin{vmatrix} q_1 & \ell_{11} \\ q_2 & \ell_{21} \end{vmatrix} + c_2 \begin{vmatrix} q_1 & \ell_{12} \\ q_2 & \ell_{22} \end{vmatrix} = 0$$

meets F at six distinct points $P_1, \ldots, P_6$ (notations as at 4.4). Let $\mathcal{F} = \mathcal{C}oker(\varphi)$ give a point in X_{40}. Performing, possibly, column operations on the matrix representing φ we may assume that $c_1 = 0, c_2 = 1$ and that P_6 is given by the equations $\ell_{12} = 0$, $\ell_{22} = 0$. Then $\mathcal{C}oker(\psi) \simeq \mathcal{O}_Z$, where Z is the union of $P_1, \ldots, P_5$. As at 4.4, $\mathcal{F}$ is an extension of $\mathcal{O}_Z$ by $\mathcal{O}_C(1)$, hence $\mathcal{F} \simeq \mathcal{O}_C(1)(P_1 + \cdots + P_5)$. Since $P_1, \ldots, P_5$ are on the irreducible conic F, no three of them are colinear.

Conversely, we must show that every sheaf of the form $\mathcal{O}_C(1)(P_1 + \cdots + P_5)$ gives a point in X_{40}. Let $F \subset \mathbb{P}^2$ be a conic curve containing $P_1, \ldots, P_5$. Because these points are assumed to be in general linear position, F is irreducible. Choose a sixth point $P_6 \in F$ distinct from the others. Let $G \subset \mathbb{P}^2$ be a cubic curve meeting F precisely at $P_1, \ldots, P_6$ (for example, the union of the three lines P_1P_2, P_3P_4, P_5P_6). Choose equations $f = 0$, $g = 0$ for F, G. Choose equations $\ell_{12} = 0$, $\ell_{22} = 0$ for P_6. We may write $f = \ell_{11}\ell_{22} - \ell_{12}\ell_{21}$, $g = q_1\ell_{22} - q_2\ell_{12}$ for some $\ell_{11}, \ell_{21} \in V^*$ and $q_1, q_2 \in \mathrm{S}^2 V^*$. Let $\psi \colon \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \to 2\mathcal{O}(-1)$ be the morphism represented by the matrix

$$\begin{bmatrix} q_1 & \ell_{11} & \ell_{12} \\ q_2 & \ell_{21} & \ell_{22} \end{bmatrix}.$$

We have $\mathcal{C}oker(\psi) \simeq \mathcal{O}_Z$, where Z is the union of $P_1, \ldots, P_5$. By construction, the maximal minors of ψ have no common factor, hence $\mathcal{K}er(\psi) \simeq \mathcal{O}(-5)$. We apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{O}_C(1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_Z \longrightarrow 0,$$

to the standard resolution of $\mathcal{O}_C(1)$ and to the resolution

$$0 \longrightarrow \mathcal{O}(-5) \longrightarrow \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \xrightarrow{\psi} 2\mathcal{O}(-1) \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We claim that the morphism $2\mathcal{O}(-1) \to \mathcal{O}_Z$ lifts to a morphism $2\mathcal{O}(-1) \to \mathcal{F}$. To see this let $\alpha \colon \mathrm{H}^0(2\mathcal{O}) \to \mathrm{H}^0(\mathcal{O}_Z)$ be the induced morphism and let $\delta \colon \mathrm{H}^0(\mathcal{O}_Z) \to \mathrm{H}^1(\mathcal{O}_C(2))$ be the connecting homomorphism associated to the exact sequence

$$0 \longrightarrow \mathcal{O}_C(2) \longrightarrow \mathcal{O}_C(2)(P_1 + \cdots + P_5) \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We must show that $\delta \circ \alpha = 0$. We will show that $\alpha^* \circ \delta^* = 0$. Taking duals in the above resolution of $\mathcal{O}_Z$ we obtain the resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \longrightarrow 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{E}xt^2(\mathcal{O}_Z, \omega_{\mathbb{P}^2}) \longrightarrow 0.$$

The induced map on global sections

$$V^* \simeq \mathrm{H}^0(\mathcal{O}(1)) \longrightarrow \mathrm{H}^0(\mathcal{E}xt^2(\mathcal{O}_Z, \omega_{\mathbb{P}^2})) \simeq \mathrm{Ext}^2(\mathcal{O}_Z, \omega_{\mathbb{P}^2}) \simeq \mathrm{H}^0(\mathcal{O}_Z)^*$$

can be identified with δ^* because, by Serre duality, δ^* is the restriction homomorphism

$$V^* \simeq \mathrm{H}^0(\mathcal{O}_C(1)) \simeq \mathrm{H}^0(\mathcal{O}_C(2)^* \otimes \omega_C) \simeq \mathrm{H}^1(\mathcal{O}_C(2))^* \longrightarrow \mathrm{H}^0(\mathcal{O}_Z)^*.$$

The induced map

$$\mathrm{Ext}^2(\mathcal{O}_Z, \omega_{\mathbb{P}^2}) \simeq \mathrm{H}^0(\mathcal{E}xt^2(\mathcal{O}_Z, \omega_{\mathbb{P}^2})) \longrightarrow \mathrm{H}^2(2\mathcal{O}(-3)) \simeq \mathrm{Ext}^2(2\mathcal{O}, \omega_{\mathbb{P}^2})$$

can be identified with α^*. It is clear now that we have $\alpha^* \circ \delta^* = 0$, proving the claim. We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-5) \longrightarrow \mathcal{O}(-5) \oplus \mathcal{O}(-3) \oplus 2\mathcal{O}(-2) \longrightarrow 2\mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Since $\mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}(1)) = 0$, the argument at 2.3.2 [9] applies to show that the morphism $\mathcal{O}(-5) \to \mathcal{O}(-5)$ above is non-zero. Canceling $\mathcal{O}(-5)$ we obtain a resolution that places $\mathcal{F}$ in X_{40}.

The inclusion $X_4 \subset \overline{X}_1$ follows from the fact that any sheaf of the form $\mathcal{O}_C(1)(P_1 + \cdots + P_5)$ as above is the limit of a sequence of sheaves $\mathcal{O}_C(1)(P_1 + \cdots + P_6 - P_7)$ as at 3.2. $\square$

Proposition 4.7. *X_4 lies in the closure of X_2.*

Proof. The argument can be found at 2.1.6 [9] and can be traced back to 3.2.3 [2]. Let $Y \subset \mathrm{M}_{\mathbb{P}^2}(6,4)$ be the subset of stable-equivalence classes of sheaves $\mathcal{G}$ satisfying the conditions $\mathrm{h}^0(\mathcal{G}(-1)) = 1$, $\mathrm{h}^0(\mathcal{G}(-2)) = 0$. We claim that for any such sheaf we have the relation $\mathrm{h}^0(\mathcal{G} \otimes \Omega^1) = 0$. To see this denote $m = \mathrm{h}^0(\mathcal{G} \otimes \Omega^1)$ and consider the Beilinson diagram (2.2.3) [2] for $\mathcal{G}(-1)$:

$$\begin{array}{ccccc} 8\mathcal{O}(-2) & \xrightarrow{\varphi_1} & (m+10)\mathcal{O}(-1) & \xrightarrow{\varphi_2} & 3\mathcal{O}\ . \\ & & & & \\ 0 & & m\mathcal{O}(-1) & \xrightarrow{\varphi_4} & \mathcal{O} \end{array}$$

As φ_4 is injective, we have $m = 0$ or 1. If $m = 1$, then $\mathcal{C}oker(\varphi_4) \simeq \mathcal{O}_L$ for a line $L \subset \mathbb{P}^2$. The exact sequence (2.2.5) [2] reads

$$0 \longrightarrow \mathcal{K}er(\varphi_1) \longrightarrow \mathcal{O}_L \longrightarrow \mathcal{G}(-1) \longrightarrow \mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1) \longrightarrow 0.$$

The map $\mathcal{O}_L \to \mathcal{G}(-1)$ is zero because $\mathrm{p}(\mathcal{O}_L) > \mathrm{p}(\mathcal{G}(-1))$ and both sheaves are semi-stable. Thus $\mathcal{O}_L \simeq \mathcal{K}er(\varphi_1)$, which is absurd.

Using the Beilinson monad for $\mathcal{G}(-1)$ we see that Y is parametrised by an open subset M inside the space of monads

$$0 \longrightarrow 8\mathcal{O}(-1) \xrightarrow{A} 10\mathcal{O} \oplus \mathcal{O}(1) \xrightarrow{B} 3\mathcal{O}(1) \longrightarrow 0$$

satisfying $B_{12} = 0$. Consider the map $\Phi \colon M \to \mathrm{Hom}(10\mathcal{O}, 3\mathcal{O}(1))$ defined by $\Phi(A, B) = B_{11}$. Using the vanishing of $\mathrm{H}^1(\mathcal{G}(1))$ for an arbitrary sheaf $\mathcal{G}$ giving a point in Y (cf. 2.1.3 [2]), we can prove that M is smooth and that Φ has surjective differential at every point. This further leads to the conclusion that the set of monads in M whose cohomology sheaf $\mathcal{G}$ satisfies the relation $\mathrm{h}^1(\mathcal{G}) = 1$ is included in the closure of the set of monads for which $\mathrm{h}^1(\mathcal{G}) = 0$. Thus X_4^{D} lies in the relative closure of $X_2^{\mathrm{D}} \cup X_3^{\mathrm{D}}$ in Y. It follows that $X_4 \subset \overline{X_2 \cup X_3}$. Since $X_3 \subset \overline{X}_2$, the conclusion follows. $\square$

5. The codimension 7 stratum

Proposition 5.1. *The sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfying the conditions $\mathrm{h}^0(\mathcal{F}(-1)) = 1$, $\mathrm{h}^1(\mathcal{F}) = 2$ are precisely the sheaves having resolution of the form*

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

where φ_{11} has linearly independent entries, $\varphi_{22} \neq 0$ and does not divide φ_{32}.

Proof. Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfy the cohomological conditions from above. Put $m = \mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1))$. Let $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$. The Beilinson monad for $\mathcal{G}$ gives the resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow 4\mathcal{O}(-2) \oplus (m+2)\mathcal{O}(-1) \longrightarrow \Omega^1 \oplus (m-3)\mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Using the Euler sequence and arguing as at 2.1.4 [9] we arrive at a resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus (m+2)\mathcal{O}(-1) \xrightarrow{\varphi} (m-3)\mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0,$$

$$\psi = \left[\begin{array}{c} 0 \\ 0 \\ \psi_{31} \end{array} \right], \qquad \varphi = \left[\begin{array}{ccc} \varphi_{11} & \varphi_{12} & 0 \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \end{array} \right].$$

Arguing as in the proof of 3.2.5 [9], we see that, modulo operations on rows and columns, ψ_{31} is represented by a matrix of the form

$$\left[\begin{array}{ccccccc} X & Y & Z & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & X & Y & Z & \cdots \end{array} \right]^{\mathrm{T}}.$$

Thus $m \geq 4$. From the fact that $\mathcal{G}$ maps surjectively onto $\mathcal{C}oker(\varphi_{11}, \varphi_{12})$ we get the reverse inequality. Thus $m = 4$, $\mathcal{C}oker(\psi_{31}) \simeq 2\Omega^1(1)$ and we obtain a resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\Omega^1(1) \xrightarrow{\varphi} \mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0,$$

in which $\varphi_{13} = 0$. Dually, we have the resolution

$$0 \longrightarrow 5\mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow 2\Omega^1 \oplus \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Combining with the standard resolution of Ω^1 yields the exact sequence

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus 5\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\varphi} 6\mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

From the semi-stability of $\mathcal{F}$ we see that $\mathrm{rank}(\varphi_{12}) = 5$, cf. argument at 2.1.4 [9]. Canceling $5\mathcal{O}(-2)$ we obtain the desired resolution of $\mathcal{F}$. The conditions imposed on φ follow from the semi-stability of $\mathcal{F}$.

Conversely, we assume that $\mathcal{F}$ has a resolution as in the proposition and we must show that there are no destabilising subsheaves. From the snake lemma we get an extension

$$0 \longrightarrow \mathcal{J}_Z(2) \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where $\mathcal{J}_Z \subset \mathcal{O}_C$ is the ideal sheaf of a zero-dimensional subscheme Z of length 2 inside a sextic curve C and $\mathbb{C}_x$ is the structure sheaf of a point. Let $\mathcal{F}' \subset \mathcal{F}$ be a subsheaf of

multiplicity at most 5, let $\mathcal{C}$ be its image in $\mathbb{C}_x$ and $\mathcal{K} = \mathcal{F}' \cap \mathcal{J}_Z(2)$. With the notations of 4.4 we have

$$\begin{aligned} \mathrm{P}_{\mathcal{F}'}(t) &= \mathrm{P}_{\mathcal{K}}(t) + \mathrm{h}^0(\mathcal{C}) \\ &= \mathrm{P}_{\mathcal{A}}(t) - \mathrm{h}^0(\mathcal{A}/\mathcal{K}) + \mathrm{h}^0(\mathcal{C}) \\ &= (6-d)t + \frac{d^2 - 7d + 6}{2} - \mathrm{h}^0(\mathcal{A}/\mathcal{K}) + \mathrm{h}^0(\mathcal{C}) \end{aligned}$$

for some integer d, $1 \le d \le 5$, hence

$$\mathrm{p}(\mathcal{F}') = \frac{1-d}{2} + \frac{\mathrm{h}^0(\mathcal{C}) - \mathrm{h}^0(\mathcal{A}/\mathcal{K})}{6-d} \le \frac{1-d}{2} + \frac{1}{6-d} < \mathrm{p}(\mathcal{F}).$$

We conclude that $\mathcal{F}$ is stable. □

Let $\mathbb{W}_5 = \mathrm{Hom}(2\mathcal{O}(-3) \oplus \mathcal{O}(-1), \mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}(1))$ and let $W_5 \subset \mathbb{W}_5$ be the set of morphisms φ from proposition 5.1. Let

$$G_5 = (\mathrm{Aut}(2\mathcal{O}(-3) \oplus \mathcal{O}(-1)) \times \mathrm{Aut}(\mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O}(1)))/\mathbb{C}^*$$

be the natural group acting by conjugation on $\mathbb{W}_5$. Let $X_5 \subset \mathrm{M}_{\mathbb{P}^2}(6,2)$ be the set of stable-equivalence classes of sheaves of the form $\mathcal{C}oker(\varphi)$, $\varphi \in W_5$.

Proposition 5.2. *There exists a geometric quotient of W_5 by G_5 and it is isomorphic to a proper open subset inside a fibre bundle with fibre $\mathbb{P}^{24}$ and base $\mathbb{P}^2 \times Y$, where Y is the Hilbert scheme of zero-dimensional subschemes of $\mathbb{P}^2$ of length 2. Moreover, W_5/G_5 is isomorphic to X_5.*

Proof. The construction of W_5/G_5 is entirely analogous to the construction of the quotient at 3.2.3 [9].

Let $\mathcal{F}$ give a point in X_5 and let $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$. The Beilinson tableau (2.2.3) [2] for $\mathcal{G}$ takes the form

$$4\mathcal{O}(-2) \xrightarrow{\varphi_1} 4\mathcal{O}(-1) \xrightarrow{\varphi_2} \mathcal{O}\ .$$

$$2\mathcal{O}(-2) \xrightarrow{\varphi_3} 6\mathcal{O}(-1) \xrightarrow{\varphi_4} 5\mathcal{O}$$

As at 6.5 [10], we have $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-4)$ and $\mathcal{K}er(\varphi_2)/\mathcal{I}m(\varphi_1) \simeq \mathcal{O}_Z$ for a scheme $Z \subset \mathbb{P}^2$ of dimension zero and length 2. The exact sequence (2.2.5) [2] takes the form

$$0 \longrightarrow \mathcal{O}(-4) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

As at 3.2.5 [9], we have $\mathcal{C}oker(\varphi_3) \simeq 2\Omega^1(1)$. This, together with (2.2.4) [2], gives the resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus 2\Omega^1(1) \longrightarrow 5\mathcal{O} \longrightarrow \mathcal{C}oker(\varphi_5) \longrightarrow 0.$$

We claim that $\mathcal{G}(1)$ has a global section which maps to a global section of $\mathcal{O}_Z$ that generates this sheaf as an $\mathcal{O}_{\mathbb{P}^2}$-module. To show this we argue as at 4.3. By 2.1.3 [2], the group $\mathrm{H}^1(\mathcal{G}(1))$ vanishes, hence we have $\mathrm{h}^0(\mathcal{G}(1)) = 10$. Since $\mathrm{h}^0(\mathcal{C}oker(\varphi_5)(1)) = 9$, we

see that $\mathcal{G}(1)$ has a global section mapping to a non-zero section s of $\mathcal{O}_Z$. We have an exact sequence

$$0 \longrightarrow \mathbb{C}_{z_1} \longrightarrow \mathcal{O}_Z \longrightarrow \mathbb{C}_{z_2} \longrightarrow 0,$$

where z_1, z_2 are not necessarily distinct points in $\mathbb{P}^2$. If s maps to zero in $\mathbb{C}_{z_2}$, then s generates $\mathbb{C}_{z_1}$. Let $\mathcal{F}'$ be the preimage of $\mathbb{C}_{z_1}$ in $\mathcal{F}$. We can apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{C}oker(\varphi_5) \longrightarrow \mathcal{F}' \longrightarrow \mathbb{C}_{z_1} \longrightarrow 0,$$

to the above resolution of $\mathcal{C}oker(\varphi_5)$ and to the standard resolution of $\mathbb{C}_{z_1}$ tensored with $\mathcal{O}(-1)$. We obtain the exact sequence

$$0 \longrightarrow \mathcal{O}(-3) \xrightarrow{\psi} \mathcal{O}(-4) \oplus 2\mathcal{O}(-2) \oplus 2\Omega^1(1) \longrightarrow \mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

We have $\mathrm{h}^1(\mathcal{F}') = \mathrm{h}^2(\mathcal{C}oker(\psi)) = 3$, hence $\mathrm{h}^0(\mathcal{F}') = 4$. On the other hand, $\mathrm{H}^0(\mathcal{C}oker(\psi))$ vanishes, hence $\mathrm{h}^0(\mathcal{F}') \geq 5$. This is absurd, so an exact sequence as above cannot exist. Thus the image of s in $\mathbb{C}_{z_2}$ is non-zero and the claim follows as at 4.3. We can now combine the resolutions of $\mathcal{O}_Z$ and of $\mathcal{C}oker(\varphi_5)$ from above to get the exact sequence

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\Omega^1(1) \longrightarrow \mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.$$

The map $\mathcal{O}(-4) \to \mathcal{O}(-4)$ is non-zero because $\mathrm{h}^1(\mathcal{G}) = 1$. We may cancel $\mathcal{O}(-4)$ to get the resolution

$$0 \longrightarrow \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\Omega^1(1) \longrightarrow \mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.$$

We saw at 5.1 how this leads to a morphism $\varphi \in W_5$ such that $\mathcal{F} \simeq \mathcal{C}oker(\varphi)$. We conclude, as at 3.1.6 [2], that the canonical bijective map $W_5/G_5 \to X_5$ is an isomorphism. □

Proposition 5.3. *The generic sheaves from X_5 are precisely the non-split extension sheaves*

$$0 \longrightarrow \mathcal{J}_Z(2) \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

where $\mathcal{J}_Z \subset \mathcal{O}_C$ is the ideal sheaf of a zero-dimensional scheme Z of length 2 inside a sextic curve $C \subset \mathbb{P}^2$ and $\mathbb{C}_x$ is the structure sheaf of a point $x \in \mathbb{P}^2$ that is not in the support of Z.

There is a dense open subset of X_5 consisting of the isomorphism classes of all sheaves of the form $\mathcal{O}_C(2)(P_1 - P_2 - P_3)$, where $C \subset \mathbb{P}^2$ is a smooth sextic curve and P_1, P_2, P_3 are distinct points on C. In particular, X_5 lies in the closure of X_3 and also in the closure of X_4.

Proof. Let $\mathcal{F} = \mathcal{C}oker(\varphi)$ give a point in X_5, where φ is a morphism as at 5.1. Let $x \in \mathbb{P}^2$ be the point given by the ideal generated by the entries of φ_{11}, let $Z \subset \mathbb{P}^2$ be the subscheme given by the equations $\varphi_{22} = 0$, $\varphi_{32} = 0$ and let $C \subset \mathbb{P}^2$ be the curve given by the equation $\det(\varphi) = 0$. We saw at 5.1 that $\mathcal{F}$ is a non-split extension of $\mathbb{C}_x$ by $\mathcal{J}_Z(2)$. Let $X_{50} \subset X_5$ be the open subset given by the condition that x be not a subscheme of Z. To show that every extension as in the proposition gives a point in X_{50} we combine the resolutions

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow 2\mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \longrightarrow \mathbb{C}_x \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{I}_Z(2) \longrightarrow \mathcal{J}_Z(2) \longrightarrow 0.$$

Here $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2}$ is the ideal sheaf of Z. We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \mathcal{O}(-4) \oplus 2\mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{I}_Z(2) \longrightarrow \mathcal{F} \longrightarrow 0.$$

The group $\mathrm{Ext}^1(\mathbb{C}_x, \mathcal{I}_Z(2))$ vanishes because x is not in Z, so we can apply the argument at 2.3.2 [9] to deduce that the morphism $\mathcal{O}(-4) \to \mathcal{O}(-4)$ in the above complex is non-zero. Canceling $\mathcal{O}(-4)$ we obtain the resolution

$$0 \longrightarrow 2\mathcal{O}(-3) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{I}_Z(2) \longrightarrow \mathcal{F} \longrightarrow 0,$$

which shows that $\mathcal{F}$ gives a point in X_{50}.

Clearly every sheaf of the form $\mathcal{O}_C(2)(P_1 - P_2 - P_3)$ is the limit of a sequence of sheaves of the form $\mathcal{O}_C(2)(-Q_1 - Q_2 - Q_3 + Q_4 + Q_5)$ as at 4.2 (make Q_1 converge to Q_4). Thus $X_5 \subset \overline{X}_3$. To prove that $X_5 \subset \overline{X}_4$ fix a sheaf $\mathcal{F} = \mathcal{O}_C(2)(Q_1 - P_2 - P_3)$ in X_5. Choosing $\mathcal{F}$ general enough, we may assume that Q_1, P_2, P_3 are non-colinear and that the line P_2P_3 meets C at six distinct points $P_2, P_3, Q_2, Q_3, Q_4, Q_5$. Then

$$\mathcal{O}_C(2)(Q_1 - P_2 - P_3) \simeq \mathcal{O}_C(1)(Q_1 + \cdots + Q_5).$$

Clearly, we can choose five distinct points R_i on C converging to Q_i, $1 \le i \le 5$, such that no three among them are colinear. According to proposition 4.6, $\mathcal{O}_C(1)(R_1 + \cdots + R_5)$ gives a point in X_4. Thus $\mathcal{O}_C(2)(Q_1 - P_2 - P_3)$ lies in the closure of X_4. □

6. The codimension 9 stratum

Proposition 6.1. *The sheaves $\mathcal{F}$ in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ satisfying the condition $\mathrm{h}^1(\mathcal{F}(1)) > 0$ are precisely the sheaves with resolution of the form*

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O} \xrightarrow{\varphi} 2\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$\varphi = \left[\begin{array}{cc} f_1 & \ell_1 \\ f_2 & \ell_2 \end{array} \right],$$

where ℓ_1, ℓ_2 are linearly independent one-forms. These sheaves are precisely the sheaves $\mathcal{J}_x(2)$, where $\mathcal{J}_x \subset \mathcal{O}_C$ is the ideal sheaf of a point x on a sextic curve $C \subset \mathbb{P}^2$.

Proof. This statement follows by duality from [10], proposition 6.1. □

Let $\mathbb{W}_6 = \mathrm{Hom}(\mathcal{O}(-4) \oplus \mathcal{O}, 2\mathcal{O}(1))$ and let $W_6 \subset \mathbb{W}_6$ be the set of morphisms φ from 6.1. Let

$$G_6 = (\mathrm{Aut}(\mathcal{O}(-4) \oplus \mathcal{O}) \times \mathrm{Aut}(2\mathcal{O}(1)))/\mathbb{C}^*$$

be the natural group acting by conjugation on $\mathbb{W}_6$. Let $X_6 \subset \mathrm{M}_{\mathbb{P}^2}(6,2)$ denote the set of stable-equivalence classes of sheaves of the form $\mathcal{C}oker(\varphi)$, $\varphi \in W_6$.

Proposition 6.2. *There exists a geometric quotient W_6/G_6, which is isomorphic to the universal sextic $\Sigma \subset \mathbb{P}^2 \times \mathbb{P}(\mathrm{S}^6\, V^*)$. Moreover, W_6/G_6 is isomorphic to X_6, so this is a smooth closed subvariety of $\mathrm{M}_{\mathbb{P}^2}(6,2)$ of codimension 9.*

Proof. For the first part of the proposition we notice, as at 3.2 [2] or at 3.2.5 [9], that the map $W_6 \to \Sigma$ defined by

$$\left[\begin{array}{cc} f_1 & \ell_1 \\ f_2 & \ell_2 \end{array} \right] \longrightarrow (x, \langle f_1\ell_2 - f_2\ell_1 \rangle),$$

x being given by the equations $\ell_1 = 0$, $\ell_2 = 0$, is a geometric quotient map. The canonical morphism $\rho \colon W_6 \to X_6$, $\rho(\varphi) = [\mathcal{C}oker(\varphi)]$, determines a bijective morphism

$$\upsilon \colon \Sigma \longrightarrow X_5, \qquad \upsilon(x, \langle f \rangle) = [\mathcal{J}_x(2)],$$

where $\mathcal{J}_x \subset \mathcal{O}_C$ is the ideal sheaf of x on the curve C given by the equation $f = 0$. As at 6.5 [10], in order to show that υ^{-1} is a morphism, we need to construct the pair (x, C) starting from $\mathrm{E}^1(\mathcal{J}_x(2))$. For technical reasons we will work, instead, with $\mathrm{E}^1(\mathcal{J}_x^{\mathrm{D}})$. Denote $\mathcal{G} = \mathcal{J}_x^{\mathrm{D}}$ and notice that $\mathcal{G}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6, 10)$ and is an extension of the form

$$0 \longrightarrow \mathcal{O}_C(3) \longrightarrow \mathcal{G} \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

The Beilinson tableau (2.2.3) [2] for $\mathcal{G}$ has the form

$$\begin{array}{lll} 2\mathcal{O}(-2) \xrightarrow{\varphi_1} \mathcal{O}(-1) & & 0 \quad . \\ & & \\ 6\mathcal{O}(-2) \xrightarrow{\varphi_3} 15\mathcal{O}(-1) & \xrightarrow{\varphi_4} & 10\mathcal{O} \end{array}$$

Since $\mathcal{G}$ is semi-stable and maps surjectively onto $\mathcal{C}oker(\varphi_1)$ we see that $\mathcal{C}oker(\varphi_1) \simeq \mathbb{C}_y$ for a point $y \in \mathbb{P}^2$ and $\mathcal{K}er(\varphi_1) \simeq \mathcal{O}(-3)$. The exact sequence (2.2.5) [2] reads

$$0 \longrightarrow \mathcal{O}(-4) \xrightarrow{\varphi_5} \mathcal{C}oker(\varphi_4) \longrightarrow \mathcal{G} \longrightarrow \mathbb{C}_y \longrightarrow 0.$$

Denote $\mathcal{G}' = \mathcal{C}oker(\varphi_5)$. The exact sequence (2.2.4) [2] yields the resolution

$$0 \longrightarrow 6\mathcal{O}(-2) \xrightarrow{\psi'} \mathcal{O}(-4) \oplus 15\mathcal{O}(-1) \xrightarrow{\varphi'} 10\mathcal{O} \longrightarrow \mathcal{G}' \longrightarrow 0.$$

We have $\mathrm{h}^0(\mathcal{G}') = 10$, hence $\mathrm{H}^0(\mathcal{G}') = \mathrm{H}^0(\mathcal{G})$. The global sections of $\mathcal{G}$ generate $\mathcal{O}_C(3)$ and $\mathcal{G}'$ is generated by its global sections. Thus $\mathcal{G}' = \mathcal{O}_C(3)$. The maximal minors of any matrix representing φ' generate the ideal of C because the Fitting support of $\mathcal{G}'$ is C. It is clear that $x = y$. In conclusion, we have obtained the pair $(x, C) \in \Sigma$ from $\mathrm{E}^1(\mathcal{G})$ by performing algebraic operations. □

Proposition 6.3. *X_6 lies in the closure of X_5.*

Proof. Any generic sheaf $\mathcal{O}_C(2)(-P)$ in X_6, with $C \subset \mathbb{P}^2$ a smooth sextic curve and $P \in C$, is the limit of a sequence of sheaves of the form $\mathcal{O}_C(2)(P_1 - P_2 - P)$ as at 5.3 (make P_1 converge to P_2). □

7. The moduli space is the union of the strata

In this final section we shall prove that $\mathrm{M}_{\mathbb{P}^2}(6, 2)$ is the union of the locally closed subsets $X_1, \ldots, X_6$ we found above.

Proposition 7.1. *There are no sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6, 2)$ and satisfying the conditions $\mathrm{h}^0(\mathcal{F}(-1)) = 0$, $\mathrm{h}^1(\mathcal{F}) = 2$.*

Proof. Assume that there is such a sheaf $\mathcal{F}$. Put $m = \mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1))$. The Beilinson monad for the dual sheaf $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$ gives the resolution

$$0 \longrightarrow 2\mathcal{O}(-2) \longrightarrow 4\mathcal{O}(-2) \oplus (m+2)\mathcal{O}(-1) \xrightarrow{\varphi} m\mathcal{O}(-1) \oplus 4\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Since $\mathcal{G}$ maps surjectively onto $\mathcal{C}oker(\varphi_{11})$, we have $m \le 3$. The rest of the proof is exactly as at 3.1.3 [9]. Let $\psi\colon 2\mathcal{O}(-2) \to (m+2)\mathcal{O}(-1)$ denote the morphism occurring in the above complex. In the case $m = 3$, say, there are three possible canonical forms for ψ given at loc.cit., each leading to a contradiction. □

Proposition 7.2. *There are no sheaves $\mathcal{F}$ giving points in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfying the cohomological conditions*

$$\mathrm{h}^0(\mathcal{F}(-1)) \le 1, \qquad \mathrm{h}^1(\mathcal{F}) \ge 3, \qquad \mathrm{h}^1(\mathcal{F}(1)) = 0.$$

Proof. The argument is the same as at 7.2 [10] with notational differences only. Assume that $\mathcal{F}$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfies the conditions $\mathrm{h}^0(\mathcal{F}(-1)) = 0$, $\mathrm{h}^1(\mathcal{F}) \ge 3$. Write $p = \mathrm{h}^1(\mathcal{F})$, $m = \mathrm{h}^0(\mathcal{F} \otimes \Omega^1(1))$. The Beilinson free monad for $\mathcal{F}$ reads

$$0 \longrightarrow 4\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \longrightarrow (m+2)\mathcal{O}(-1) \oplus (p+2)\mathcal{O} \xrightarrow{\psi} p\mathcal{O} \longrightarrow 0,$$
$$\psi = \left[\begin{array}{cc} \eta & 0 \end{array}\right],$$

and yields a resolution

$$0 \longrightarrow 4\mathcal{O}(-2) \oplus m\mathcal{O}(-1) \xrightarrow{\varphi} \mathcal{K}er(\eta) \oplus (p+2)\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0$$

in which $\varphi_{12} = 0$. We have $m + 2 - p = \operatorname{rank}(\mathcal{K}er(\eta)) \le 4$ because φ is injective. Thus

$$\mathrm{h}^0(\mathcal{F}(1)) = 3(p+2) + \mathrm{h}^0(\mathcal{K}er(\eta)(1)) - m \ge 2(p+2) \ge 10$$

forcing $\mathrm{h}^1(\mathcal{F}(1)) \ge 2$. Assume, instead, that $\mathrm{h}^0(\mathcal{F}(-1)) = 1$. The Beilinson monad for the dual sheaf $\mathcal{G} = \mathcal{F}^{\mathrm{D}}(1)$ reads

$$0 \longrightarrow p\mathcal{O}(-2) \longrightarrow (p+2)\mathcal{O}(-2) \oplus (m+2)\mathcal{O}(-1) \longrightarrow m\mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{O} \longrightarrow 0$$

and leads to a resolution

$$0 \longrightarrow p\mathcal{O}(-2) \longrightarrow \mathcal{O}(-3) \oplus (p-1)\mathcal{O}(-2) \oplus (m+2)\mathcal{O}(-1) \xrightarrow{\varphi}$$
$$(m-3)\mathcal{O}(-1) \oplus 5\mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

in which $\varphi_{13} = 0$. Since $\mathcal{G}$ maps surjectively onto $\mathcal{C}oker(\varphi_{11}, \varphi_{12})$, we have $m - 3 \le p$. Dualising the above resolution we get a monad for $\mathcal{F}$ of the form

$$0 \longrightarrow 5\mathcal{O}(-2) \oplus (m-3)\mathcal{O}(-1) \longrightarrow (m+2)\mathcal{O}(-1) \oplus (p-1)\mathcal{O} \oplus \mathcal{O}(1) \xrightarrow{\psi} p\mathcal{O} \longrightarrow 0,$$
$$\psi = \left[\begin{array}{ccc} \eta & 0 & 0 \end{array}\right].$$

The exact sequence

$$0 \longrightarrow 5\mathcal{O}(-2) \oplus (m-3)\mathcal{O}(-1) \longrightarrow \mathcal{K}er(\eta) \oplus (p-1)\mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0$$

gives the estimate

$$\mathrm{h}^0(\mathcal{F}(1)) = 3(p-1) + 6 + \mathrm{h}^0(\mathcal{K}er(\eta)(1)) - (m-3) \ge 3p + 6 - m \ge 2p + 3 \ge 9.$$

We deduce that $\mathrm{h}^1(\mathcal{F}(1)) \ge 1$. □

Proposition 7.3. *Let $\mathcal{F}$ be a sheaf giving a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfying the condition $\mathrm{h}^1(\mathcal{F}(1)) = 0$. Then $\mathrm{h}^0(\mathcal{F}(-1)) = 0$ or 1.*

Proof. Let $\mathcal{F}$ give a point in $\mathrm{M}_{\mathbb{P}^2}(6,2)$ and satisfy the condition $\mathrm{h}^0(\mathcal{F}(-1)) \geq 2$. As at 2.1.3 [2], there is an injective morphism $\mathcal{O}_C \to \mathcal{F}(-1)$ for a curve $C \subset \mathbb{P}^2$. This curve has degree 5 or 6, otherwise $\mathcal{O}_C$ would destabilise $\mathcal{F}(-1)$. Assume that $\deg(C) = 5$. The quotient sheaf $\mathcal{C} = \mathcal{F}/\mathcal{O}_C(1)$ has Hilbert polynomial $\mathrm{P}(t) = t + 2$ and zero-dimensional torsion $\mathcal{T}$ of length at most 1 (the pull-back in $\mathcal{F}$ of $\mathcal{T}$ would be a destabilising subsheaf if its length were at least 2). If $\mathcal{T} = 0$, then $\mathcal{C} \simeq \mathcal{O}_L(1)$ for a line $L \subset \mathbb{P}^2$. We get that $\mathrm{h}^0(\mathcal{F}(-1)) = 2$ and that the morphism $\mathcal{O}(1) \to \mathcal{O}_L(1)$ lifts to a morphism $\mathcal{O}(1) \to \mathcal{F}$. The horseshoe lemma leads to the resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O} \longrightarrow 2\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Thus $\mathrm{h}^1(\mathcal{F}(1)) = 1$. Assume now that $\mathcal{T}$ has length 1. Let $\mathcal{F}' \subset \mathcal{F}$ be the pull-back of $\mathcal{T}$. According to 3.1.5 [9], we have $\mathrm{h}^0(\mathcal{F}'(-1)) = 1$. Since $\mathcal{F}/\mathcal{F}' \simeq \mathcal{C}/\mathcal{T} \simeq \mathcal{O}_L$ for a line $L \subset \mathbb{P}^2$, we get $\mathrm{h}^0(\mathcal{F}(-1)) = 1$, contradicting our choice of $\mathcal{F}$.

Assume now that C is a sextic curve. The quotient sheaf $\mathcal{T} = \mathcal{F}/\mathcal{O}_C(1)$ is zero-dimensional of length 5. Let $\mathcal{T}' \subset \mathcal{T}$ be a subsheaf of length 4 and let $\mathcal{F}'$ be its preimage in $\mathcal{F}$. We claim that $\mathcal{F}'$ gives a point in $\mathrm{M}_{\mathbb{P}^2}(6,1)$. If this were not the case, then $\mathcal{F}'$ would have a destabilising subsheaf $\mathcal{F}''$, which may be assumed to be semi-stable. By proposition 2.3, $\mathcal{F}$ is stable, so we have the inequalities $1/6 < \mathrm{p}(\mathcal{F}'') < 1/3$. This leaves only two possibilities: that $\mathcal{F}''$ give a point in $\mathrm{M}_{\mathbb{P}^2}(5,1)$ or in $\mathrm{M}_{\mathbb{P}^2}(4,1)$. In the first case $\mathcal{F}/\mathcal{F}''$ is isomorphic to the structure sheaf of a line, hence $\mathrm{h}^0(\mathcal{F}(-1)) = \mathrm{h}^0(\mathcal{F}''(-1)) = 0$ or 1, cf. [9]. This contradicts our choice of $\mathcal{F}$. In the second case $\mathcal{F}/\mathcal{F}''$ is easily seen to be semi-stable, hence it is isomorphic to the structure sheaf of a conic curve. We get $\mathrm{h}^0(\mathcal{F}(-1)) = \mathrm{h}^0(\mathcal{F}''(-1)) = 0$, cf. [2], contradicting our choice of $\mathcal{F}$. This proves the claim, i.e. that $\mathcal{F}'$ is semi-stable. We have $\mathrm{h}^0(\mathcal{F}'(-1)) \geq 1$ so, according to [10], there are two possible resolutions for $\mathcal{F}'$:

$$0 \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{F}' \longrightarrow 0$$

or

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-1) \longrightarrow \mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Assume that $\mathcal{F}'$ has the first resolution. We apply the horseshoe lemma to the extension

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

to the given resolution of $\mathcal{F}'$ and to the resolution

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

As $\mathrm{h}^0(\mathcal{F}'(-1)) = 1$ and $\mathrm{h}^0(\mathcal{F}(-1)) \geq 2$, the morphism $\mathcal{O}(1) \to \mathbb{C}_x$ lifts to a morphism $\mathcal{O}(1) \to \mathcal{F}$. We obtain the resolution

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow 2\mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2\mathcal{O} \longrightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus 2\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

The morphism $\mathcal{O}(-1) \to 2\mathcal{O}(-3) \oplus \mathcal{O}(-2)$ occurring above is zero and $\mathrm{Ext}^1(\mathbb{C}_x, \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1)) = 0$, so we can argue as at 2.3.2 [9] to conclude that $\mathcal{F}$ is a trivial extension of $\mathbb{C}_x$ by $\mathcal{F}'$. This contradicts the semi-stability of $\mathcal{F}$. Assume, finally, that $\mathcal{F}'$ has the second resolution. We can apply the horseshoe lemma as above, leading to the resolution

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-1) \oplus 2\mathcal{O} \longrightarrow \mathcal{O} \oplus 2\mathcal{O}(1) \longrightarrow \mathcal{F} \longrightarrow 0.$$

We see from this that $\mathrm{h}^1(\mathcal{F}(1)) = 1$. □

References

[1] J.-M. Drézet, *Variétés de modules alternatives.* Ann. Inst. Fourier **49** (1999), 57-139.

[2] J.-M. Drézet and M. Maican, *On the geometry of the moduli spaces of semi-stable sheaves supported on plane quartics*. Geom. Dedicata **152** (2011), 17–49.

[3] J.-M. Drézet and G. Trautmann, *Moduli spaces of decomposable morphisms of sheaves and quotients by non-reductive groups.* Ann. Inst. Fourier **53** (2003), 107-192.

[4] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves.* Aspects of Mathematics E31, Vieweg, Braunschweig, 1997.

[5] A. King, *Moduli of representations of finite dimensional algebras.* Q. J. Math. Oxf. II Ser. **45** (1994), 515-530.

[6] J. Le Potier, *Faisceaux semi-stables de dimension* 1 *sur le plan projectif.* Rev. Roumaine Math. Pures Appl. **38** (1993), 635-678.

[7] M. Maican, *On two notions of semistability.* Pacific J. Math. **234** (2008), 69-135.

[8] ———, *A duality result for moduli spaces of semistable sheaves supported on projective curves.* Rend. Sem. Mat. Univ. Padova **123** (2010), 55-68.

[9] ———, *On the moduli spaces of semi-stable plane sheaves of dimension one and multiplicity five.* arXiv:1007.1815

[10] ———, *On the moduli space of semi-stable plane sheaves with Euler characteristic one and supported on sextic curves.* arXiv:1105.0112

[11] D. Mumford, J. Fogarty and F. Kirwan. *Geometric invariant theory.* 3rd enl. ed. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3 Folge 34. Springer Verlag, Berlin, 1993.

[12] V. Popov and E. Vinberg. *Invariant theory* in *Algebraic geometry IV*, A. Parshin and I. Shafarevich (eds.), G. Kandall (transl.) Encyclopaedia of mathematical sciences v. 55. Springer Verlag, Berlin, 1994.

Mario Maican, Institute of Mathematics of the Romanian Academy, Calea Griviţei 21, 010702 Bucharest, Romania

E-mail address: mario.maican@imar.ro